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Modern cosmologies from empty Kaluza-Klein solutions in 5D

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ABSTRACT: We show that the empty five-dimensional solutions of Davidson-Sonnenschein-Vozmediano, *Phys. Rev. D* **32** (1985) 1330, in the “old” Kaluza-Klein gravity, under appropriate interpretation can generate an ample variety of cosmological models in 4D, which include the higher-dimensional modifications to general relativity predicted by “modern” versions of noncompactified 5D gravity as, e.g., induced-matter and braneworld theories. This is the first time that these solutions are investigated in a systematic way as embeddings for cosmological models in 4D. They provide a different formulation, which is complementary to the approaches used in current versions of 5D relativity.

KEYWORDS: Cosmology of Theories beyond the SM, Classical Theories of Gravity

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1 Introduction

The concept that our universe might be a 4-dimensional hypersurface embedded in a higher-dimensional universe constitutes an extremely provocative idea, from a theoretical, philosophical and practical point of view.

From a theoretical point of view, the well-known Campbell’s theorem [1, 2] serves as a ladder to go between manifolds whose dimensionality differs by one. This theorem, which is valid in any number of dimensions, implies that every solution of the 4D Einstein equations with arbitrary energy-momentum tensor can be embedded, at least locally, in a solution of the 5D Einstein field equations in vacuum. It is the backbone of induced-matter theory (IM) [3], and brings to fruition the philosophy of geometrodynamics [4] in which “matter and charge may be manifestations of the topology of space”¹

From a practical point of view, extensions of general relativity to five and more dimensions seem to provide the best route to unification of gravity with interactions of particle

¹As Feynman put it: “It would indeed be very beautiful to have $G^{\mu}_{\nu} = 0$ everywhere, so that, in words used recently to describe geometrodynamics, matter comes from no matter, and charge comes from no charge” [4].

physics [5]–[8]. Braneworld theory (BW) proposes a model where our spacetime is a singular hypersurface embedded in an empty (no matter sources) 5-dimensional anti-de Sitter space [9], which might provide a possible solution to the hierarchy problem between weak and Planck scales.

One important physical problem in higher-dimensional theories is to develop a full understanding of implications in 4D. Therefore, it is essential to compare and contrast the effective pictures generated in 4D by different versions of 5-dimensional relativity, like, e.g., the classical Kaluza-Klein theory with “cylinder condition” (KK) and the two above-mentioned approaches, where the extra dimension is not assumed to be compactified.

Today, it is well known that IM and BW originate the same effective 4-dimensional world, despite of the fact that they have different motivation and interpretation [10]. For example, in cosmological applications, on *every* 4D hypersurface orthogonal to the extra dimension IM reproduces, although in different conventions and notation, the generalized (or modified) Friedmann equation of BW on a \mathbf{Z}_2 -symmetric brane.

The question arises of whether one can establish some connection between cosmological models constructed under similar conditions in KK, IM, and BW. The aim of this work is to study this question. In particular, we ask whether models in the “old” Kaluza-Klein theory can be made compatible with IM and BW, i.e., with the concept that our universe is a 4D hypersurface embedded in a 5-dimensional world.

We will see that the answer to this question is positive. In our discussion we concentrate our attention to a family of KK cosmological models first discovered by Davidson, Sonnenschein and Vozmediano [11]. These models share with IM and BW the property that they are solutions to the Einstein field equations in an empty 5-dimensional space

$${}^{(5)}G_{AB} = k_{(5)}^2 \Lambda_{(5)} \gamma_{AB}, \tag{1.1}$$

where $A, B = 0, 1, 2, 3, 4$; $k_{(5)}^2$ is a constant introduced for dimensional considerations, and γ_{AB} is the extended 5D Friedmann-Robertson-Walker (FRW) metric

$$dS^2 = \gamma_{AB} dx^A dx^B = dt^2 - A^2(t) d\Sigma_k^2 - \Phi^2(t) dy^2. \tag{1.2}$$

Here $d\Sigma_k^2$ is the metric on the unit three dimensional plane ($k = 0$), hyperboloid ($k = -1$) or sphere $k = 1$, viz.,

$$d\Sigma_k^2 = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{1.3}$$

In our notation, the Davidson-Sonnenschein-Vozmediano (DSV) solutions are given by²

$$A^2(t) = \begin{cases} c_1 \cosh \omega t + c_2 \sinh \omega t + \frac{2k}{\omega^2}, & \text{for } k_{(5)}^2 \Lambda_{(5)} = \frac{3\omega^2}{2}, \\ -kt^2 + c_1 t + c_2, & \text{for } k_{(5)}^2 \Lambda_{(5)} = 0, \\ c_1 \cos \omega t + c_2 \sin \omega t - \frac{2k}{\omega^2}, & \text{for } k_{(5)}^2 \Lambda_{(5)} = -\frac{3\omega^2}{2}, \end{cases} \tag{1.4}$$

²Some typos in the original paper [11] are fixed here.

$$\Phi(t) = B_0 \left(\frac{dA}{dt} \right), \tag{1.5}$$

where c_1, c_2, B_0 are constants of integration with the appropriate dimensions.³ To study the effective 4-dimensional world, these authors used the Kaluza-Klein ansatz, which consists in assuming that the extra dimension is compactified (rolled up to a small size), and identifying

$$g_{\mu\nu} = \Phi \gamma_{\mu\nu}, \tag{1.6}$$

with the metric of the effective 4-dimensional world [12]. An observer in 4D, who is not aware of the existence of an extra dimension, interprets the metric $g_{\mu\nu}$ as if it were governed by an effective 4D energy-momentum tensor. For the DSV solutions, the effective isotropic pressure p^{eff} and density ρ^{eff} satisfy the “equation of state”

$$\rho^{\text{eff}} = p^{\text{eff}} + \frac{k_{(5)}^2 \Lambda_{(5)}}{4\pi G \Phi}, \tag{1.7}$$

which for $\Lambda_{(5)} = 0$ reduces to the familiar stiff equation.

However, this is not the only way to establish the effective 4-dimensional picture. Other alternatives for dimensional reduction are formulated in induced-matter and braneworld theories, where the extra dimension is not assumed to be compactified. Since the DSV solutions are vacuum solutions, it is of theoretical interest to reanalyze their 4-dimensional interpretation from the perspective of these theories.

In section 2, we sketch the main features of noncompactified theories. We show that, by means of a simple transformation $t \leftrightarrow y$, from (1.4)–(1.5) one can generate a family of static 5-dimensional solutions. Then, within the context of IM in the comoving frame, we demonstrate that the DSV solutions and their static counterparts lead to significantly different scenarios in 4D. In section 3, we study the 4-dimensional picture measured by an observer who, instead of being at rest, is moving in a DSV universe. We find that such an observer can perceive a rich variety of cosmological scenarios, including cosmological models where the induced matter satisfies the barotropic equation of state. In section 4, we show that the static DSV models allow us to reproduce the modified, or generalized, Friedmann cosmological equation of braneworld models [13]–[15].

2 Relaxing the Kaluza-Klein ansatz

Modern theories of gravity in 5D introduce two important new ingredients. First, they do not require the extra dimension to be compact; in principle it can be infinitely large. Second, a large extra dimension can be either spacelike or timelike; both are physically admissible (see, e.g., [16] and references therein). In this regard, one should be careful to discriminate between temporal (spatial) dimensions, which actually have physical units of time (length); and timelike (spacelike) ones, which merely have timelike (spacelike) signature [17].

³ B_0 has dimensions of length. In the first and third solutions c_1 and c_2 are dimensionless. In the second solution c_1 has dimensions of $(\text{length})^{-1}$ and c_2 is dimensionless

2.1 Static solutions

An immediate consequence of these new “ingredients” is that the metric

$$dS^2 = dt^2 - A^2(t)d\Sigma_k^2 + \epsilon B_0^2 \left(\frac{dA}{dt} \right)^2 dy^2, \quad (2.1)$$

with timelike extra dimension ($\epsilon = +1$), is also a physically valid solution of the field equations $G_{AB} = k_{(5)}^2 \Lambda_{(5)} \gamma_{AB}$. In this case making the transformation $t \leftrightarrow y$ and $k \rightarrow \epsilon k$, from (1.4)–(1.5) we obtain the following set of *static* solutions

$$dS^2 = \gamma_{CD} dx^C dx^D = B_0^2 \left(\frac{dA}{dy} \right)^2 dt^2 - A^2 d\Sigma_k^2 + \epsilon dy^2, \quad (2.2)$$

where

$$A^2(y) = \begin{cases} c_1 \cosh \omega y + c_2 \sinh \omega y + \frac{2\epsilon k}{\omega^2}, & k_{(5)}^2 \Lambda_{(5)} = \frac{3\epsilon \omega^2}{2} \\ -\epsilon k y^2 + c_1 y + c_2, & \text{for } k_{(5)}^2 \Lambda_{(5)} = 0, \\ c_1 \cos \omega y + c_2 \sin \omega y - \frac{2\epsilon k}{\omega^2} & \text{for } k_{(5)}^2 \Lambda_{(5)} = -\frac{3\epsilon \omega^2}{2}. \end{cases} \quad (2.3)$$

We will show in sections 3 and 4 that these solutions, which we will call static DSV solutions, may be used to establish a connection between the DSV solutions, the FRW models of conventional 4D relativity, and the modified Friedmann equation. Although we are not especially promoting timelike extra dimensions, because they can lead to closed timelike curves (CTC) and hence allow causality violation,⁴ for the sake of generality in our discussion we keep $\epsilon = \pm 1$.

2.2 Effective gravity in 4D

The effective metric measured by an observer depends on her/his state of motion. The simplest physical scenario emerges in the rest (also called *comoving*) frame, which in the present case means ($dx^i = dy = 0$). In such a frame, the spacetime is recovered by going onto some hypersurface $\Sigma_{y_0} : y = y_0 = \text{constant}$, which is orthogonal to the unit 5D vector $\hat{n}_A = \Phi \delta_A^4$ tangent to the extra coordinate. The effective equations for gravity in Σ_{y_0} are obtained from dimensional reduction of the 5-dimensional Einstein field equations, which is based on Campbell’s theorem. It consists in isolating the 4D part of the relevant 5D geometric quantities and use them to construct the 4D Einstein tensor ${}^{(4)}G_{\alpha\beta}$. For the 5D metric $dS^2 = \gamma_{\mu\nu} dx^\mu dx^\nu + \epsilon \Phi^2 dy^2$ the result is⁵

$${}^{(4)}G_{\alpha\beta} = 8\pi G {}^{(4)}T_{\alpha\beta} \equiv \frac{2}{3} k_{(5)}^2 \left[{}^{(5)}T_{\alpha\beta} + ({}^{(5)}T_4^4 - \frac{1}{4} {}^{(5)}T) g_{\alpha\beta} \right] - \epsilon \left(K_{\alpha\lambda} K_\beta^\lambda - K_\lambda^\lambda K_{\alpha\beta} \right) + \frac{\epsilon}{2} g_{\alpha\beta} \left(K_{\lambda\rho} K^{\lambda\rho} - (K_\lambda^\lambda)^2 \right) - \epsilon E_{\alpha\beta} \quad (2.4)$$

⁴It has been argued that physics can be compatible with CTC [18, 19]. For theories with timelike extra dimensions see, e.g., [20]–[28].

⁵There are five more equations; in the present case they reduce to $E_\mu^\mu = 0$ and ${}^{(4)}T_{\nu;\mu}^\mu = 0$.

where ${}^{(4)}G_{\alpha\beta}$ is calculated with the 4D metric⁶ $g_{\mu\nu} = \gamma_{\mu\nu}$; ${}^{(4)}T_{\alpha\beta}$ is the effective energy-momentum tensor (EMT) measured in Σ_{y_0} ; ${}^{(5)}T_{AB}$ is the EMT in 5D; $\epsilon = \pm 1$ depending on whether the extra dimension is spacelike or timelike; $K_{\mu\nu}$ is the extrinsic curvature of Σ_{y_0} ,

$$K_{\alpha\beta} = \frac{1}{2}\mathcal{L}_{\hat{n}}g_{\alpha\beta} = \frac{1}{2\Phi}\frac{\partial g_{\alpha\beta}}{\partial y}; \tag{2.5}$$

$E_{\mu\nu}$ is the projection of the 5D Weyl tensor ${}^{(5)}C_{ABCD}$ orthogonal to \hat{n}^A , i.e., ‘‘parallel’’ to Σ_{y_0} , viz.,

$$E_{\alpha\beta} = {}^{(5)}C_{\alpha A\beta B}\hat{n}^A\hat{n}^B = -\frac{1}{\Phi}\frac{\partial K_{\alpha\beta}}{\partial y} + K_{\alpha\rho}K_{\beta}^{\rho} - \epsilon\frac{\Phi_{\alpha;\beta}}{\Phi}, \tag{2.6}$$

and $\Phi_{\alpha} \equiv \partial\Phi/\partial x^{\alpha}$. In what follows we denote ${}^{(4)}T_0^0 \equiv \rho^{\text{eff}}$ and ${}^{(4)}T_1^1 = {}^{(4)}T_2^2 = {}^{(4)}T_3^3 \equiv -p^{\text{eff}}$.

2.3 Interpretation of DSV solutions on Σ_{y_0}

We now apply the above expressions to the original DSV solutions. Since the metric in (1.4)–(1.5) is independent of y , the extrinsic curvature $K_{\alpha\beta}$ of hypersurfaces $y = y_0$ is identically zero. Considering that $E_{\mu\nu}$ is traceless, the effective matter in Σ_{y_0} can be interpreted as a mixture of vacuum fluid and (Weyl) radiation. For future purposes we give the explicit form of the induced matter quantities in Σ_{y_0} :

$$\rho^{\text{eff}} = \rho + \frac{\Lambda_{(4)}}{8\pi G}, \quad p^{\text{eff}} = p - \frac{\Lambda_{(4)}}{8\pi G}, \quad \Lambda_{(4)} = \frac{3\epsilon_{\Lambda}\omega^2}{4}, \quad p = \frac{\rho}{3}, \tag{2.7}$$

where $\epsilon_{\Lambda} = (1, 0, -1)$ for $\Lambda_{(5)} > 0$, $\Lambda_{(5)} = 0$ and $\Lambda_{(5)} < 0$, respectively, and

$$8\pi G\rho = \frac{3f_{\Lambda}}{4A^4(t)}, \quad f_{\Lambda} = \begin{cases} \omega^{-2} [4k^2 - \omega^4(c_1^2 - c_2^2)], & \Lambda_{(5)} > 0, \\ c_1^2 + 4kc_2, & \Lambda_{(5)} = 0, \\ \omega^{-2} [\omega^4(c_1^2 + c_2^2) - 4k^2], & \Lambda_{(5)} < 0. \end{cases} \tag{2.8}$$

In cosmological applications, this interpretation seems to be more satisfactory than the one given by the equation of state (1.7), derived from the Kaluza-Klein ansatz. We note that, in general, ρ is not necessarily positive, except for $k = 0$ and $\Lambda_{(5)} \leq 0$. However, if we add the initial condition $A(0) = 0$ then $\rho \geq 0$ in all three cases, regardless of the choice of k .

Let us now turn our attention to the static solutions (2.2)–(2.3). On every hypersurface Σ_{y_0} the metric functions in (2.2) are constants, and the line element is Minkowskian. However, the components of the extrinsic curvature (2.5) are not zero, and the matter variables do not vanish in general. In fact, in all three cases, the effective matter induced in Σ_{y_0} is given by

$$p^{\text{eff}} = -\frac{\rho^{\text{eff}}}{3}, \quad 8\pi G\rho^{\text{eff}} = \frac{3k}{\mathcal{A}^2(y_0)}. \tag{2.9}$$

⁶Various versions of IM can be found in the literature for different definitions of the physical metric in 4D (see, e.g., [29] and references therein).

Thus, the relationship between the effective quantities is similar to the equation of state $\rho = -3p$ for “nongravitating matter”, which has been discussed in a number of different contexts [5, 30]–[33]. The nonvanishing components of the Riemann tensor on this hypersurface are

$$R_{1313} = R_{1212} \sin^2 \theta = \frac{1 - kr^2}{r^2} R_{2323} = -\frac{k\mathcal{A}^2(y_0)r^2 \sin^2 \theta}{1 - kr^2}. \quad (2.10)$$

Thus, on Σ_{y_0} the effective spacetime is empty and Riemann-flat only for $k = 0$. A similar, but not identical, *non-vacuum Minkowskian* spacetime is discussed in [34].

3 Generating spacetime on a dynamical hypersurface

Thus, when we identify our spacetime with some hypersurface $y = y_0 = \text{constant}$ in a DSV universe, the effective matter in Σ_{y_0} is restricted to behave either as a radiation-like fluid or as nongravitating matter. Both scenarios are unsatisfactory if we desire to obtain more general cosmologies. The problem is that the embedding $y = y_0$ is too rigid.

A more “flexible” approach, that respects the spatial homogeneity and isotropy of FRW models, is to consider that 4D observers are at rest only in 3D ($dx^i = 0$), but moving in 5D. That is we relax the condition $dy = 0$ of section 2.2 and assume that $y = y(t)$, or in parametric form

$$t = S(\tau), \quad y = Y(\tau), \quad (3.1)$$

where τ is the proper time. In this approach our spacetime is recovered on a dynamical 4D hypersurface, which we will denote as $\Sigma_{Y(\tau)}$, with local coordinates (τ, r, θ, ϕ) . In this section we will see that an observer living in $\Sigma_{Y(\tau)}$, who is unaware of the motion through an empty 5D-dimensional universe, will interpret the expansion or contraction of the universe as if it were governed by an effective matter satisfying some equation of state, which is not necessarily restricted to be $\rho = 3p$ or $\rho = -3p$.

We consider a foliation of the 5D-dimensional manifold such that (3.1) is itself a hypersurface of the foliation. Then, for the 5D cosmological line element

$$d\mathcal{S}^2 = N^2(t, y)dt^2 - A^2(t, y)d\Sigma_k^2 + \epsilon\Phi^2(t, y)dy^2, \quad (3.2)$$

the metric induced on every hypersurface of the foliation is (hereafter $\dot{X} \equiv dX/d\tau$; $X_t \equiv dX/dt$; $X' \equiv dX/dy$)

$$ds^2 = \left[N^2(t, y)\dot{S}^2 + \epsilon\Phi^2(t, y)\dot{Y}^2 \right] d\tau^2 - a^2(\tau)d\Sigma_k^2. \quad (3.3)$$

On $\Sigma_{Y(\tau)}$ this metric has to take the usual FRW form. Therefore we require

$$\left[N^2(t, y)\dot{S}^2 + \epsilon\Phi^2(t, y)\dot{Y}^2 \right]_{t=S(\tau), y=Y(\tau)} = 1, \quad a(\tau) = A(t, y)|_{t=S(\tau), y=Y(\tau)}, \quad (3.4)$$

which ensures that τ is the proper time. It also shows that the functions S and Y are not independent, meaning that the state of movement of $\Sigma_{Y(\tau)}$ is parameterized just by

one function of the proper time τ . The unit vector \hat{n}^A normal to the foliation, and the four-velocity vector u^A tangent to the foliation can be written as

$$\begin{aligned} \hat{n}_A &= \frac{s}{\sqrt{1+\epsilon V^2}}(-VN, 0, 0, 0, \Phi), & \hat{n}^A &= \frac{s}{\sqrt{1+\epsilon V^2}}\left(-\frac{V}{N}, 0, 0, 0, \frac{\epsilon}{\Phi}\right), \\ u_A &= \frac{1}{\sqrt{1+\epsilon V^2}}(N, 0, 0, 0, \epsilon\Phi V), & u^A &= \frac{1}{\sqrt{1+\epsilon V^2}}\left(\frac{1}{N}, 0, 0, 0, \frac{V}{\Phi}\right). \end{aligned} \quad (3.5)$$

Here $\hat{n}_A \hat{n}^A = \epsilon$; $u_A u^A = 1$; $n_A u^A = 0$; $s = \pm 1$ determines the orientation of the normal,⁷ and V is the *coordinate* velocity of $\Sigma_{Y(\tau)}$. This follows from the fact that the displacement $\hat{n}_A dx^A$ must vanish on every hypersurface of the foliation. Therefore,

$$V = \frac{\Phi dy}{N dt} = \frac{\Phi \dot{Y}}{N \dot{S}} = \frac{\Phi \dot{Y}}{\sqrt{1 - \epsilon \Phi^2 \dot{Y}^2}}, \quad (3.6)$$

where we have used (3.4) and assumed $\dot{S} > 0$.

3.1 Interpretation of DSV solutions on $\Sigma_{Y(\tau)}$

We now apply the above general formulate to elucidate what kind of 4D cosmological models can be obtained as projections of DSV solutions on a moving hypersurface $\Sigma_{Y(\tau)}$.

3.1.1 Time-dependent DSV solutions

First, let us show that the effective matter induced on $\Sigma_{Y(\tau)}$ is no longer pure Weyl radiation as it occurs on Σ_{y_0} , (2.7)–(2.8). In fact, the extrinsic curvature of the surface $\Sigma_{Y(\tau)}$ is in general non-zero, because $\hat{n}^0 \sim V \neq 0$. In particular, for the DSV solutions (1.4)–(1.5)

$$K_0^0 = -s B_0 A_t \left[\frac{\ddot{Y}}{\dot{S}} + \frac{A_{tt} \dot{Y}}{A_t} \right], \quad K_1^1 = K_2^2 = K_3^3 = -\frac{s B_0 A_t^2 \dot{Y}}{A}, \quad (3.7)$$

K_{04} and K_{44} are obtained from $K_{AB} \hat{n}^B = 0$. Substituting in (2.4) we find that, apart from the Weyl radiation given by $E_{\mu\nu}$, there are a number of additional terms which do not cancel out, but vanish when $\dot{Y} = 0$ ($V = 0$). This is a general result valid for any 5D metric with no-dependence of the extra y coordinate. It uncovers the misleading nature of the notion, frequently found in the literature, that “radiation is the only kind of matter one can obtain in the induced-matter interpretation as long as the cylinder condition⁸ is in place.”

We now proceed to develop a general expression for the energy density induced on $\Sigma_{Y(\tau)}$. To this end we use (3.4), which in the present case reads

$$\left[\dot{S}^2 - B_0^2 A_t^2 \dot{Y}^2 \right]_{|_{t=S(\tau)}} = 1, \quad a(\tau) = A(S(\tau)); \quad (3.8)$$

obtain S as a function of $a(\tau)$ from (1.4); calculate $\dot{S} = \dot{a}(dS/da)$, and substitute $\dot{a}^2 = 8\pi G \rho^{\text{eff}} a^2/3 - k$. The result is

$$B_0^2 A_t^2 \dot{Y}^2 = \dot{S}^2 - 1 = \frac{4[8\pi G \rho^{\text{eff}} - 3\epsilon_\Lambda \omega^2/4]a^4(\tau) - 3f_\Lambda}{3\delta_\Lambda}, \quad \delta_\Lambda \geq 0, \quad (3.9)$$

⁷For $s = 1$ ($s = -1$) the displacement $n_A dx^A$, for fixed t , is in the increasing (decreasing) direction of y .

⁸Kaluza’s cylinder condition sets the derivatives with respect to the additional coordinate to zero.

where ϵ_Λ and f_Λ are the coefficients defined in (2.7)–(2.8), and

$$\delta_\Lambda = \begin{cases} f_\Lambda - 4ka^2 + \omega^2 a^4, & k_{(5)}\Lambda_{(5)} = \frac{3\omega^2}{2}, \\ f_\Lambda - 4ka^2, & \Lambda_{(5)} = 0, \\ f_\Lambda - 4ka^2 - \omega^2 a^4, & k_{(5)}\Lambda_{(5)} = -\frac{3\omega^2}{2}. \end{cases} \quad (3.10)$$

The condition $\delta_\Lambda \geq 0$ is necessary to ensure that $S(\tau)$ is a real function. In principle, f_Λ can have either sign, but models where a can reach $a = 0$ require $f_\Lambda \geq 0$. Finally, from (1.5), (3.6) and (3.8) we get the desired formula, viz.,

$$8\pi G\rho^{\text{eff}} = \Lambda_{(4)} + \frac{3f_\Lambda}{4a^4} + \frac{V^2}{1-V^2} \left(\frac{3\delta_\Lambda}{4a^4} \right). \quad (3.11)$$

We note that when $V = 0$ we recover (2.7)–(2.8). The last term is always positive because $|V| < 1$ ($\dot{Y}^2 \geq 0$), and in general is not radiation-like because δ_Λ is a function of a .

In principle, one can find the motion of $\Sigma_{Y(\tau)}$ for any given equation of state. The algorithm is as follows: from the field equations find $a(\tau)$; use (1.4) to get S as a function of $a(\tau)$; substitute into (3.8) to obtain $V(\tau)$ and $Y(\tau)$; finally, the range of applicability of the model is set by the condition $\dot{Y}^2 \geq 0$.

As an illustration, let us consider the equation of state $p^{\text{eff}} = w\rho^{\text{eff}}$ with $w = \text{constant}$, which implies $\rho \sim 1/a^{3(1+w)}$. Substituting this into (3.9) we find that $\dot{Y}^2 \geq 0$ is satisfied, in the whole range of a , if $(\Lambda_{(5)} = 0, w = 1/3)$ or $(\Lambda_{(5)} < 0, w \geq 1/3)$; in any other case the applicability is restricted to certain portions of the evolution. For example, in the case of spatially flat models with $\Lambda_{(4)} = 0$ for which $a(\tau) \sim \tau^{2/3(w+1)}$, we find $S = C\tau^{4/3(w+1)}$, and⁹

$$V^2 = 1 - \frac{9(w+1)^2 \tau^{2(3w-1)/3(w+1)}}{16C^2}. \quad (3.12)$$

The model works well ($0 \leq V^2 \leq 1$) either in the very early universe ($1/3 < w \leq 1$) or at “late” times ($0 \leq w < 1/3$). An interesting by-product of the discussion is that $V = \text{constant}$ for $w = 1/3$. Namely, $V = 0$ for pure Weyl (geometric) radiation, while $V = \text{constant} \neq 0$ for a mixture of Weyl radiation, photons and ultra-relativistic matter.

Another approach for constructing 4D cosmological models from (1.4)–(1.5) consists in prescribing the motion of $\Sigma_{Y(\tau)}$. The problem with this is that we have no physical arguments in support of any particular choice. The only criterion seems to be “mathematical simplicity”. Here we just show an interesting representative example, which arises from the assumption

$$S(\tau) = \tau + C\tau^{4/3(w+1)}, \quad (3.13)$$

with $0 \leq \tau < \infty$ and $C = \text{constant} > 0$. In this case $\dot{Y}^2 \geq 0$ ($|V| \leq 1$) in the whole range of τ , for all $w \in (-1, 1]$. In particular, for $w = 0$, $\dot{S}|_{\tau \rightarrow 0} \rightarrow 1$ in the early universe, which by virtue of (3.8) implies $Y|_{\tau \rightarrow 0} \rightarrow \text{constant}$, i.e., in the early universe $\Sigma_{Y(\tau)} \rightarrow \Sigma_{y_0}$. Consequently, at early times the model reduces to a radiation filled universe analogous to the one considered in (2.7), while at late times the effective matter on $\Sigma_{Y(\tau)}$ behaves like *cosmological dust*. Similar cosmologies can be generated on $\Sigma_{Y(\tau)}$ from the other solutions in (1.4).

⁹Please note that V is *not* the speed of light c . Thus $V = 1$ does not mean that $\Sigma_{Y(\tau)}$ is light-like.

3.1.2 Static DSV solutions

We now consider the case where our spacetime is a dynamical hypersurface moving in the static background (2.2)–(2.3). Here, in addition to the freedom in the choice of $Y(\tau)$ (or $S(\tau)$), we also have a freedom in the signature of the extra dimension. In this case the functions S and Y must satisfy

$$B_0^2 \mathcal{A}'^2 \dot{S}^2 + \epsilon \dot{Y}^2 = 1, \quad a(\tau) \equiv \mathcal{A}(Y(\tau)), \quad (3.14)$$

which follow from (3.4), and assure that the spacetime metric induced in $\Sigma_{Y(\tau)}$ has the usual FRW form. The matter induced on $\Sigma_{Y(\tau)}$ is determined by the Einstein equations, namely

$$\begin{aligned} \frac{8\pi G}{3} \rho^{\text{eff}} &= \frac{\mathcal{A}'^2 \dot{Y}^2}{\mathcal{A}^2} + \frac{k}{\mathcal{A}^2}, \\ 4\pi G \left(p^{\text{eff}} + \frac{\rho^{\text{eff}}}{3} \right) &= -\frac{\mathcal{A}'' \dot{Y}^2}{\mathcal{A}} - \frac{\mathcal{A}' \ddot{Y}}{\mathcal{A}}. \end{aligned} \quad (3.15)$$

The last equation shows that $\rho^{\text{eff}} = -3p^{\text{eff}}$, when $Y = \text{constant}$. However, there is another solution, namely $\mathcal{A}(Y(\tau)) \sim \tau$, which generates Milne’s universe (the corresponding Y and S are obtained from (2.3) and (3.14), respectively). For any other choice the effective matter in $\Sigma_{Y(\tau)}$ behaves as “regular” gravitating matter.

Again, we can derive a general expression for ρ^{eff} analogous to (3.11). Although we omit the formulae, for all three solutions in (2.3) we find that the matter induced on a dynamical hypersurface is compatible with the barotropic equation of state, in the whole range of τ , and $w \neq 1/3$, provided $\Lambda_{(5)} \leq 0$, which is equivalent to $\epsilon = -1$ and $\epsilon = 1$ in the hyperbolic and trigonometric solution, respectively. For $w = 1/3$, the extra dimension can be either spacelike or timelike. As an illustration, let us consider spatially flat models with $\Lambda_{(5)} = 0$. In this case

$$a^2(\tau) = c_1 Y(\tau) + c_2 = (C_1 \tau + C_2)^{4/3(w+1)}, \quad (3.16)$$

where C_1 and C_2 are constants of integration. Substituting into (3.14) we get

$$\frac{B_0^2 c_1^2}{4} \dot{S}^2 = a^2 - \frac{16\epsilon C_1^2 a^{3(1-w)}}{9(1+w)^2 c_1^2}. \quad (3.17)$$

Since $\dot{S}^2 > 0$, and $0 \leq a(\tau) < \infty$, this expression shows that the extra dimension must be spacelike ($\epsilon = -1$) for any $w \neq 1/3$. For $w = 1/3$, it can be either spacelike or timelike, depending on the choice of the constants.

Another simple example that demonstrates the diversity of cosmological scenarios, arises from the observation that setting $\dot{S} = 1$ in (3.8) requires $Y = Y_0$, which gives back the conventional models with radiation-like effective matter (2.7). In static DSV solutions the assumption $\dot{S} = 1$ is inconsistent with $Y = Y_0$, but generates an ample variety of collapsing and bouncing cosmological models. In fact, the scale factor resulting from the integration of (3.14) can be written in the parametric form

$$a = \frac{B_0 c_1}{2} \sin \eta, \quad \tau - \tau_0 = \frac{B_0^2 c_1}{4} \left(\eta - \frac{1}{2} \sin 2\eta \right), \quad \text{for } \epsilon = -1, \quad (3.18)$$

where τ_0 is a constant of integration, which can be set equal to zero. Thus, despite the fact that the universe is spatially flat, for $\epsilon = -1$ it recollapses in a finite proper time: $a(\tau)$ grows from zero at $\tau = 0$ ($\eta = 0$) to a maximum value $a = B_0 c_1/2$, which is reached when $\tau = B_0^2 c_1 \pi/8$ ($\eta = \pi/2$), and decreases again to zero at $\tau = B_0^2 c_1 \pi/4$. For small values of a , we assume $\eta \ll 1$. Then $a \sim \eta$, $\tau \sim \eta^3$, so that $a \sim \tau^{1/3}$, which corresponds to a fluid with a stiff equation of state, i.e., $p^{\text{eff}} = \rho^{\text{eff}}$.

In a similar way, for $\epsilon = 1$ the solution can be written as

$$a = \frac{B_0 c_1}{2} \cosh \eta, \quad \tau - \tau_0 = \frac{B_0^2 c_1}{4} \left(\eta + \frac{1}{2} \sinh 2\eta \right), \quad \text{for } \epsilon = 1. \quad (3.19)$$

Here the scale factor changes monotonically, increasing from $a = B_0 c_1/2$, at $\tau = 0$ ($\eta = 0$), to infinity for $\tau \rightarrow \infty$ ($\eta \rightarrow \infty$). For large values of τ we find $a \sim \tau^{1/2}$, which corresponds to radiation, $\rho^{\text{eff}} = 3p^{\text{eff}}$.

To sum up, in this section we have shown that an observer riding in a hypersurface $\Sigma_{Y(\tau)}$ can perceive a rich variety of cosmological scenarios, including cosmological models where the induced matter satisfies the barotropic equation of state, not only radiation-like or nongravitating matter.

4 Our universe as a moving brane in a static DSV universe

In the last section, within the context of IM, we have obtained cosmological models in 4D as projections of the 5-dimensional DSV solutions on a moving hypersurface $\Sigma_{Y(\tau)}$. The aim of this section is to extend the discussion to embrace the so-called braneworld cosmological models. In these models gravity propagates in all 5-dimensions, whereas particles and fields are confined to a singular 4D hypersurface (the brane), thereby accounting for their relatively greater strength.

The metric g_{AB} induced on the hypersurfaces of the foliation, which is defined by the orthonormal vectors (3.5), is given by

$$g_{AB} = \gamma_{AB} - \epsilon \hat{n}_A \hat{n}_B, \quad g_{AB} \hat{n}^B = 0. \quad (4.1)$$

Thus, using (3.2) and (3.5) we obtain

$$\begin{aligned} g_{00} &= \frac{N^2}{1 + \epsilon V^2} & g_{ij} &= \gamma_{ij}, \\ g_{04} &= \epsilon \left(\frac{V}{N} \right) g_{00}, & g_{44} &= \left(\frac{V}{N} \right)^2 g_{00}, \end{aligned} \quad (4.2)$$

where, from (2.2)

$$N \equiv B_0 \left(\frac{dA}{dy} \right). \quad (4.3)$$

The extrinsic curvature is

$$\begin{aligned}
 K_{00} &= s \frac{\epsilon N^2}{(1 + \epsilon V^2)^{3/2}} \left[\frac{N'}{N} - \frac{\epsilon V V'}{1 + \epsilon V^2} \right], \\
 K_1^1 &= K_2^2 = K_3^3 = s \frac{\epsilon \mathcal{A}'}{\mathcal{A} \sqrt{1 + \epsilon V^2}}, \\
 K_{04} &= \epsilon \left(\frac{V}{N} \right) K_{00}, \quad K_{44} = \left(\frac{V}{N} \right)^2 K_{00},
 \end{aligned} \tag{4.4}$$

with trace

$$K = \frac{s\epsilon}{\sqrt{1 + \epsilon V^2}} \left[\frac{N'}{N} + \frac{3\mathcal{A}'}{\mathcal{A}} - \frac{\epsilon V V'}{1 + \epsilon V^2} \right]. \tag{4.5}$$

In the present case the coordinate velocity (3.6) reduces to $V = \dot{Y}/\sqrt{1 - \epsilon \dot{Y}^2}$. Therefore, in the above equations

$$\sqrt{1 + \epsilon V^2} = \frac{1}{\sqrt{1 - \epsilon \dot{Y}^2}}, \quad V V' = \frac{\ddot{Y}}{(1 - \epsilon \dot{Y}^2)^2}, \tag{4.6}$$

where we have used that $V' = \dot{V}/\dot{Y}$ [34].

4.1 Matter on the brane

The metric g_{AB} is continuous across the brane but there is a jump in the extrinsic curvature ($K_{AB|s=1} = -K_{AB|s=-1}$). Israel's boundary conditions [35] relate this jump to the brane energy-momentum tensor, viz.,

$$T_{AB} + \sigma g_{AB} = -\frac{2\epsilon}{k_{(5)}^2} (K_{AB} - g_{AB} K), \tag{4.7}$$

where $K_{AB} = K_{AB|s=+1}$; σ is the tension of the brane, which is interpreted as the vacuum energy density, and T_{AB} represents the energy-momentum tensor of ordinary matter in the brane. For perfect fluid it is

$$T_{AB} = (\rho + p)u_A u_B - p g_{AB}, \tag{4.8}$$

where ρ and p are the energy density and isotropic pressure measured by an observer with velocity u^A .

If we substitute (2.3), (4.3), (4.6) into (4.7)–(4.8), and set $y = Y(\tau)$, then we obtain the braneworld matter in terms of the embedding function $Y(\tau)$, and its first and second derivatives. However, in cosmological models, what we need is to find explicit expressions relating the matter in 4D to the dynamics of the scale factor $a(\tau)$. To accomplish this goal we have to specify Y as a function of $a(\tau)$.

1. From the first solution in (2.3) we obtain

$$\cosh \omega Y = \frac{c_1 [\omega^2 a^2(\tau) - 2k\epsilon] - c_2 \Delta_h}{\omega^2 (c_1^2 - c_2^2)}, \tag{4.9}$$

with $\Delta_h = \sqrt{[\omega^2 a^2(\tau) - 2k\epsilon]^2 - \omega^4 (c_1^2 - c_2^2)}$. The expression for $\sinh \omega Y$ is obtained by changing $c_1 \leftrightarrow c_2$ in (4.9), but keeping Δ_h . We note that $\cosh \omega Y$ and $\sinh \omega Y$ remain finite for $c_1 = c_2$.

2. From the second solution in (2.3) we get

$$Y(\tau) = \frac{\epsilon}{2k} \left\{ c_1 - \sqrt{c_1^2 + 4\epsilon k [c_2 - a^2(\tau)]} \right\}, \quad k \neq 0. \quad (4.10)$$

For $k = 0$, $Y(\tau) = [a^2(\tau) - c_2]/c_1$.

3. Finally, we note that the third solution in (2.3) is formally obtained from the first one under the substitution $\omega \rightarrow i\omega$, $c_2 \rightarrow ic_2$. Therefore we have

$$\cos \omega Y = \frac{c_1 [\omega^2 a^2(\tau) + 2k\epsilon] - c_2 \Delta_t}{\omega^2 (c_1^2 + c_2^2)}, \quad (4.11)$$

where $\Delta_t = \sqrt{\omega^4 (c_1^2 + c_2^2) - [\omega^2 a^2(\tau) + 2k\epsilon]^2}$. The expression for $\sin \omega Y$ is obtained from this equation by changing $c_1 \rightarrow c_2$, $c_2 \rightarrow -c_1$.

Substituting (4.9)–(4.11) into the original equations for braneworld matter in terms of $Y(\tau)$, we obtain (to simplify the presentation we omit cumbersome intermediate calculations)

$$3 \left(\frac{\dot{a}}{a} \right)^2 = \begin{cases} \frac{3\epsilon\omega^2}{4} - \frac{\epsilon k_{(5)}^4}{12} (\rho + \sigma)^2 - \frac{3k}{a^2} + \frac{3\epsilon [4k^2 - (c_1^2 - c_2^2)\omega^4]}{4\omega^2 a^4}, & \text{for } k_{(5)}^2 \Lambda_{(5)} = \frac{3\epsilon\omega^2}{2}, \\ -\frac{\epsilon k_{(5)}^4}{12} (\rho + \sigma)^2 - \frac{3k}{a^2} + \frac{3\epsilon}{a^4} \left(\frac{c_1^2}{4} + \epsilon k c_2 \right), & \text{for } k_{(5)}^2 \Lambda_{(5)} = 0, \\ -\frac{3\epsilon\omega^2}{4} - \frac{\epsilon k_{(5)}^4}{12} (\rho + \sigma)^2 - \frac{3k}{a^2} - \frac{3\epsilon [4k^2 - (c_1^2 + c_2^2)\omega^4]}{4\omega^2 a^4}, & \text{for } k_{(5)}^2 \Lambda_{(5)} = -\frac{3\epsilon\omega^2}{2}. \end{cases} \quad (4.12)$$

By means of direct calculation we have checked that the three models satisfy the conservation equation

$$(\dot{\rho} + \dot{\sigma}) + 3 \frac{\dot{a}}{a} (\rho + p) = 0, \quad (4.13)$$

so we need not to have an explicit expression for $(p - \sigma)$. However, it may be worth emphasizing that this conservation equation holds regardless of the choice of functional dependence of $\sigma(\tau)$, although it is usually taken as $\sigma = \sigma_0 = \text{constant}$.

4.1.1 The generalized Friedmann equation

Following a notation that now is standard, the above solutions can be written as

$$3 \left(\frac{\dot{a}}{a} \right)^2 = \Lambda_{(4)} + 8\pi G \rho - \frac{3k}{a^2} - \frac{\epsilon k_{(5)}^4}{12} \rho^2 + \frac{\mathcal{C}}{a^4} \quad (4.14)$$

where

$$8\pi G = -\frac{\epsilon k_{(5)}^4 \sigma}{6}, \quad \Lambda_{(4)} = \begin{cases} \frac{3\epsilon}{4} \left[\omega^2 - \frac{1}{9} k_{(5)}^2 \sigma^2 \right], & \text{for } k_{(5)}^2 \Lambda_{(5)} = \frac{3\epsilon\omega^2}{2}, \\ -\epsilon \frac{k_{(5)}^4 \sigma^2}{12}, & \text{for } k_{(5)}^2 \Lambda_{(5)} = 0, \\ -\frac{3\epsilon}{4} \left[\omega^2 + \frac{1}{9} k_{(5)}^2 \sigma^2 \right], & \text{for } k_{(5)}^2 \Lambda_{(5)} = -\frac{3\epsilon\omega^2}{2}. \end{cases} \quad (4.15)$$

and

$$\mathcal{C} = \begin{cases} \frac{3\epsilon[4k^2 - (c_1^2 - c_2^2)\omega^4]}{4\omega^2}, & \text{for } k_{(5)}^2 \Lambda_{(5)} = \frac{3\epsilon\omega^2}{2}, \\ \frac{3\epsilon(c_1^2 + 4\epsilon kc_2)}{4}, & \text{for } k_{(5)}^2 \Lambda_{(5)} = 0, \\ -\frac{3\epsilon[4k^2 - (c_1^2 + c_2^2)\omega^4]}{4\omega^2}, & \text{for } k_{(5)}^2 \Lambda_{(5)} = -\frac{3\epsilon\omega^2}{2}. \end{cases} \quad (4.16)$$

The first three terms in the r.h.s. of (4.14) give the standard cosmology, while the fourth and fifth terms are local and non-local higher dimensional modifications to general relativity, respectively. In particular, the term \mathcal{C}/a^4 can be interpreted as an effective (Weyl) radiation coming from the bulk to the brane. The first solution in (4.14)–(4.15) with $\epsilon = -1$ is in agreement with the well-known generalized Friedmann equation [13]–[15], which has widely been used and discussed in the braneworld literature. In this equation one can always fine-tune the parameters ω and σ in such a way as to set the 4D effective cosmological constant $\Lambda_{(4)}$ equal to zero.

However, this is not so for the second and third solutions in (4.14)–(4.15). For the solution with $\Lambda_{(5)} = 0$ this is not surprising. But, in the third solution $\Lambda_{(4)}$ cannot be set equal to zero either, despite of the fact that $\Lambda_{(5)} \neq 0$.

4.2 Consistency relations for the embedding

First, we have to verify the positivity of \dot{S}^2 . In fact, from (3.17) we have learned that \dot{S}^2 is not automatically positive; in principle the requirement $\dot{S}^2 > 0$ may lead to some specific physical restrictions. In the present case one can show that $\dot{S}^2 > 0$, for all three solutions, without imposing restrictions whatsoever.

Indeed, from (3.14) it follows that $B_0^2 \mathcal{A}'^2 \dot{S}^2 = (1 - \epsilon \dot{Y}^2)$. Now we use (4.9), (4.10), (4.11) to obtain \dot{Y} in terms of a and \dot{a} . Next, we eliminate \dot{a} in \dot{Y} by utilizing (4.12). After a long but straightforward calculation we get

$$B_0^2 \mathcal{A}'^4 \dot{S}^2 = \frac{k_{(5)}^2}{36} (\rho + \sigma)^2 a^2(\tau), \quad (4.17)$$

for the three cases, regardless of the choice of ϵ , k and \mathcal{C} .

Second, we have to make sure that $Y(\tau)$ is a real function for all values of τ . This requires the quantities under the roots in (4.9), (4.10) and (4.11) to be non-negative. Using (4.16), this condition can be expressed as

$$\begin{aligned} \omega^2 a^4 - 4k\epsilon a^2 + \frac{4\epsilon\mathcal{C}}{3} &\geq 0, & \text{for } k_{(5)}^2 \Lambda_{(5)} = \frac{3\epsilon\omega^2}{2}, \\ \epsilon [\mathcal{C} - 3ka^2] &\geq 0, & \text{for } k_{(5)}^2 \Lambda_{(5)} = 0, \\ \omega^2 a^4 + 4k\epsilon a^2 - \frac{4\epsilon\mathcal{C}}{3} &\leq 0, & \text{for } k_{(5)}^2 \Lambda_{(5)} = -\frac{3\epsilon\omega^2}{2}. \end{aligned} \quad (4.18)$$

These equations, impose geometrical constraints on the evolution of the scale factor a , for every given set $(\epsilon, k, \mathcal{C})$. They indicate what kind of braneworld cosmological models are, in principle, compatible with the embeddings under consideration. However, physical

requirements, as the energy conditions on ρ and p and $\dot{a}^2 \geq 0$, demand more stringent constraints on the evolution of the scale factor.

4.2.1 Possible cosmological scenarios

We now proceed to examine, in some detail, the 4D cosmological models allowed by (4.18) when the extra dimension is spacelike ($\epsilon = -1$).

The models with $k_{(5)}^2 \Lambda_{(5)} = -3\omega^2/2$ are specially interesting because, as we mentioned above, they correspond to the generalized Friedmann equation discussed in braneworld cosmologies [13]–[15]. In this case the inequality $\omega^2 a^4 + 4ka^2 - 4\mathcal{C}/3 \geq 0$ has several solutions corresponding to the various values of \mathcal{C} and k :

1. $k = -1$: For $-\infty < \mathcal{C} \leq -3/\omega^2$, there are no restrictions on a , i.e., the above inequality is satisfied for *all* values of a . For $-3/\omega^2 < \mathcal{C} < 0$, the allowed range of a is (assuming a positive) either

$$0 \leq a(\tau) \leq \frac{\sqrt{2}}{|\omega|} \left(1 - \sqrt{1 + \omega^2 \mathcal{C}/3}\right)^{1/2}, \quad \text{or} \quad a(\tau) \geq \frac{\sqrt{2}}{|\omega|} \left(1 + \sqrt{1 + \omega^2 \mathcal{C}/3}\right)^{1/2}. \quad (4.19)$$

For $\mathcal{C} \geq 0$, a is restricted to the region defined by the second inequality.

2. $k = (0, 1)$: For $\mathcal{C} \leq 0$, there are no restrictions on a . For $\mathcal{C} > 0$ we obtain

$$a(\tau) \geq (4\mathcal{C}/3\omega^2)^{1/4}, \quad \text{and} \quad a(\tau) \geq \frac{\sqrt{2}}{|\omega|} \left(\sqrt{1 + \omega^2 \mathcal{C}/3} - 1\right)^{1/2}, \quad (4.20)$$

for $k = 0$ and $k = 1$, respectively. We note that $a(\tau)$ can never reach zero, i.e. there is no big-bang, unless $\mathcal{C} = 0$.

Summing-up, the case where $k_{(4)}^2 \Lambda_{(5)} = -3\omega^2/2$ allows recollapsing, bouncing and ever-expanding cosmological models. It is not difficult to see that similar scenarios are possible when $\Lambda_{(5)} = 0$. However, for $k_{(4)}^2 \Lambda_{(5)} = 3\omega^2/2$ *all the models are recollapsing*.

Let us now examine the physics in the bulk. A simple inspection of (2.2)–(2.3) shows that $g_{tt}(y) = \infty$ at $\mathcal{A}(y) = 0$. To analyze this in more detail we calculate the Kretschmann scalar $I = R_{ABCD}R^{ABCD}$. We obtain

$$I = 40\alpha^2 + \frac{72\beta^2}{\mathcal{A}^8}, \quad (4.21)$$

where

$$\alpha = \left(\frac{\omega^2}{4}, 0, -\frac{\omega^2}{4}\right), \quad \beta = \frac{4\mathcal{C}}{3} = \left[\frac{\omega^4(c_1^2 - c_2^2) - 4k^2}{4\omega^2}, \frac{4kc_2 - c_1^2}{4}, -\frac{\omega^4(c_1^2 + c_2^2) - 4k^2}{4\omega^2}\right], \quad (4.22)$$

for $k_{(5)}^2 \Lambda_{(5)} = -3\omega^2/2$; $k_{(5)}^2 \Lambda_{(5)} = 0$ and $k_{(5)}^2 \Lambda_{(5)} = 3\omega^2/2$, respectively. The above suggests the introduction of the dimensionless coordinate z , viz.,

$$z = \mathcal{A}. \quad (4.23)$$

In terms of z the static DSV solutions (2.2)–(2.3), with $\epsilon = -1$, become

$$dS^2 = B_0^2 \left(k + \alpha z^2 - \frac{\beta}{z^2} \right) dt^2 - \frac{dz^2}{(k + \alpha z^2 - \beta/z^2)} - z^2 d\Sigma_k^2, \quad (4.24)$$

which is the five-dimensional analogue of the Schwarzschild-de Sitter spacetime. It shows that for $\beta = 0$ ($\mathcal{C} = 0$) the bulk is singularity-free; it is either dS₅ or AdS₅, depending on whether $\alpha < 0$ or $\alpha > 0$. However, for $\beta \neq 0$ the bulk is singular at $z = 0$. For several values of the constants, the singularity is covered by a horizon located at $z = z_h$.

For $\alpha > 0$

$$z_h^2 = \begin{cases} \sqrt{c_1^2 - c_2^2} - 2/\omega^2, & k = 1, \\ \sqrt{c_1^2 - c_2^2}, & k = 0, \\ \sqrt{c_1^2 - c_2^2} + 2/\omega^2, & k = -1. \end{cases} \quad (4.25)$$

Alternative formulas for z_h^2 can be obtained by using (4.22) for expressing $\sqrt{c_1^2 - c_2^2}$ in terms of \mathcal{C} . We find that $z_h^2 > 0$ requires $\mathcal{C} > 0$ for $k = (0, 1)$ and $\mathcal{C} > -(3/4\omega^2)$ for $k = -1$. For these parameters the scale factor $a(\tau)$ can never vanish because it is bounded from below. A similar situation occurs for $\alpha = 0$ and $k = 1$.

Our analysis shows that ever-expanding big-bang cosmological models on the effective 4D brane require $\alpha \geq 0$ ($\Lambda_{(5)} \leq 0$) and $\mathcal{C} = 0$ ($\beta = 0$), which in turn, by virtue of (4.21), demand the bulk to be free of singularities (not a black hole in 5D). In this context, the singularity in 4D can be interpreted as a consequence of the topological separation of our universe from the 5D bulk [36].

5 Summary

We have studied the classical Davidson-Sonnenschein-Vozmediano cosmological solutions, originally obtained and interpreted in the context of 5-dimensional Kaluza-Klein theory with cylindrical symmetry, where the effective metric in 4D is constructed by a factorization technique (1.6). Nowadays, such a factorization is not required and the extra dimension is not assumed to be compactified. The two versions of 5D relativity in vogue, namely induced matter theory and membrane theory, employ a 5D Kaluza-Klein type of metric but identify our spacetime with some 4D hypersurface embedded in 5D.

In section 2, by employing Campbell’s theorem, which is the fundamental mathematical support of induced-matter theories, we analyzed the physics induced on a 4D hypersurface $y = y_0 = \text{constant}$. Also, exploiting the symmetry of the 5D metrics, we constructed the static counterpart to Davidson-Sonnenschein-Vozmediano solutions. The equations of state of the matter induced in 4D is $\rho = 3p$ in the FRW case, and $\rho = -3p$ in the static case, corresponding to radiation-like and nongravitating matter, respectively.

In section 3 we considered the most general embedding compatible with spatial homogeneity and isotropy. The spacetime was identified with a dynamical hypersurface $\Sigma_{Y(\tau)}$ defined by one function of the proper time. We found that, an observer living in $\Sigma_{Y(\tau)}$, who is unaware of her/his motion through an empty 5-dimensional universe, will interpret

the scale factor $a(\tau)$ as if it were governed by an effective matter satisfying some equation of state, which is not necessarily restricted to be radiation-like or nongravitating. In fact, we discussed a number of cosmological scenarios, which include ever-expanding, collapsing and bouncing models, with different equations of state including the barotropic one.

In section 4 we used the braneworld paradigm to embed our spacetime as a dynamical 4D hypersurface in a static DSV universe. We considered three possible cases, viz., $k_{(4)}^2 \Lambda_{(5)} = 3\epsilon\omega^2/2$, $\Lambda_{(5)} = 0$, $k_{(4)}^2 \Lambda_{(5)} = -3\epsilon\omega^2/2$, which require different consistency relations for the embedding. As a consequence, although the generalized Friedmann equation in 4D, looks the same in all cases, they represent distinct physical scenarios. The most notorious difference is between the cases with $k_{(4)}^2 \Lambda_{(5)} = 3\epsilon\omega^2/2$ and $k_{(4)}^2 \Lambda_{(5)} = -3\epsilon\omega^2/2$. The first case, which for $\epsilon = -1$ gives back previous results in the literature [13]–[15], is compatible with a wide range of cosmological and $\Lambda_{(4)}$ can be set equal to zero. The second case, however, is compatible only with recollapsing models and $\Lambda_{(4)}$ cannot be set equal to zero.

In conclusion, here we have obtained a number of 4D cosmological models as projections of DSV cosmological solutions on some 4-dimensional hypersurface. This is the first time in the literature that these solutions are used in a systematic way to provide a different formulation of 4D cosmologies. Therefore, our approach is complementary to the usual formalism employed in IM [3] and BW [13]–[15], and shows that Davidson-Sonnenschein-Vozmediano solutions are compatible with the notion that our universe can be an evolving 4D hypersurface embedded in a 5-dimensional world.

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